Abstract
The Betz-Joukowsky (BJ) theory is the starting point to study wind turbines because it provides the upper bound of their power at the mega-scale. However, not all the assumptions that support this theory have been clearly identified in the literature. In this work, an upscaling approach is applied to the mass, linear momentum and mechanical energy equations at the microscopic scale in order to obtain their mega-scale counterparts. The analysis allows a systematic identification of the assumptions and restrictions involved in the BJ theory, which were found to serve as a guideline for the shape and size of the averaging domain. The derivations show that the actuator disc concept is indispensable in order to recover Froude’s theorem. In addition, numerical experiments evidence that the restrictions related to neglecting mass transport and air power at the lateral surfaces of the averaging domain are the hardest ones to satisfy. The results from this work expand the current understanding of the BJ theory and open new research lines to improve it by identifying its ancillary assumptions and restrictions.

Keywords: Betz-Joukowsky theory, wind turbines, upscaling, actuator disc.

1 Introduction

Wind energy, along with solar energy, is one of the main alternative clean resources nowadays. During the past few decades, there has been an exponential increment in the installation of wind farms across the world and in particular in Mexico. Industrial planning and scheduling relies upon the use of mathematical models applicable at the field-scale (or mega-scale from this point on).

The first attempt to model this type of system is due to Frederick Lanchester (1915), simultaneously followed by Albert Betz (1920) and Nikolái Zhukovsky (from here on, Joukowsky for consistency with occidental literature) (Joukowsky, 1920), as explained in the works by Bergey (1979) and van Kuik (2007). These three efforts have in common the use of an actuator disc to represent flow around the blades of a turbine. However, Lanchester did not predict the same value of the actuator disc velocity derived by Betz and Joukowsky (Okulov & van Kuik, 2012). The latter two authors, independently concluded that 16/27 of the air power is the upper bound that a wind turbine can extract. Consequently, this limit is strictly associated to Betz and Joukowsky (from here...
on BJ). Briefly, their theory relies on coupling the work performed by the wind over the actuator disc with the reduction of the kinetic energy.

The actuator disc concept is not restricted to the study of wind turbines; actually, it was introduced by Rankine (1865) and Froude (1889) for studying marine propellers and envisages the turbine as being a permeable rotating disc with an infinite number of blades. Recently, the use of the actuator disc concept was validated by comparison with numerical simulations as shown by Li et al. (2014). Classically, the hub, nacelle and post are not considered when the actuator disc concept is used (cf. Meyers & Meneveauy, 2010). The reader interested on a detailed analysis of this concept is referred to the recent book by van Kuik (2018). Certainly, the actuator disc concept allows for considerably simplifying the mathematical treatments at the price of losing some information about the turbine, such as its particular geometry. This subject will be further discussed later in this work.

Nowadays, the BJ theory is a mandatory starting point for any mathematical theory regarding wind turbines. For example, wake theories based upon top-hat models (Jensen, 1983) or Gaussian distributions (Prasad, 2014) for the velocity deficit profile use, as an input parameter, the downwind velocity from the BJ theory. There have been many papers that suggest modifications to make this theory closer to real wind turbines. For example, some theories have included rotational effects induced by the actuator disc, such as the work by Goldstein (1929) where the vortex theory is used in screw propellers considering two blades. A serious drawback of the above analyses (and also of the BJ theory) is that no indication about the geometry of the blades is given. This motivated Glauert (1926) to use the blade element theory (Drzewiecki, 1920), where each element of a blade is regarded as a suitable airfoil element moving in its appropriate manner so that the resulting force on a blade can be computed as the sum of all elements along it. This type of analysis allows predicting the power coefficient (i.e., the ratio between the power delivered to the wind turbine with respect to the air power) as a function of the tip-speed ratio. In this way, the BJ limit was recovered only as the tip-speed ratio tends to infinity. Furthermore, Dymet (1989) suggested incorporating losses to Betz theory in order to provide more realistic maximum power values. According to this author, the maximum power is one half of the BJ limit. A comparison of the predictions from the classical theories of Joukowski and Glauert is available from van Kuik et al. (2015) and a more extended analysis can be found in Chapter 5 of Sørensen (2016).

The above works have made contributions to wind-turbine theory by taking into account the effect of the number of blades, their geometry and the resulting wake over the predictions of the power coefficient. Moreover, other relevant contributions have been presented that rely on different approaches. Gorban’ et al. (2001) argued that the BJ theory is an overestimate because it neglects the curvature of the fluid streams and that a more accurate upper bound is about 30 percent for free fluids. A similar value (≈ 0.362) was proposed by Huleihil (2009) to be the efficiency of a wind turbine at maximum power production. Actually, it has been suggested that the 16/27 limit can be theoretically exceeded in tidal turbines (Garret & Cummins, 2007; Vennell, 2013) since they are confined in channels, at the cost of a reduction of the flow; however, this would require improvements in the structure and design of the turbines and it is an active research field nowadays.

The literature review presented so far suggests that the BJ limit is indeed a maximum and that advances in the theory should produce smaller values of the power coefficient. This point of view is not shared by some authors. In particular, Inglis (1979) suggested that this limit should be regarded as a guideline since this value can be found to be larger or smaller depending on the rotational kinetic energy of the downwind stream and turbulent mixing from the outside boundaries. Shortly after, Greet (1980) was more severe in his criticism to Betz theory and argued that the momentum equation used in the theory is incorrect and that the system of equations was unclosed since more unknowns than equations were present. This point of view was shared by Rauh & Seelert (1984), who integrated the momentum and energy equations and concluded that the momentum balance is not different from the one corresponding to the mechanical energy, thus making the set of equations incomplete with respect to the number of unknowns.

A point that all the above works have in common is the analysis of mass, linear (and in some cases also angular) momentum and mechanical energy transport at a scale level in which the dependent variables (velocity and pressure) are no longer position-dependent. In other words, the balance equations are treated in an integral form at the mesoscale. As a matter of fact, there are some works (e.g., Sørensen, 2016; van Kuik, 2018) that depart from the integral form of the balance equations, but rapidly reach the desired result without detailing the
Hierarchical system description

Before commencing with the derivations, it is pertinent to dedicate some paragraphs to describe the system and the hierarchy of characteristic lengths involved. In addition, some concepts that are not commonly used in wind turbines literature (and are borrowed from porous media systems) are clearly identified.

In many environmental systems, it is natural to identify certain levels of scale, which are linked to spatial variations of state properties (Gray & Gray, 2016). Certainly, such separation of length scales can be applied to wind turbines systems. In Figure 1, three levels of scale can be identified, namely: 1) The microscale is the smallest scale level where spatial variations of momentum transport are appreciated and it will be denoted as $\ell_{mi}$. Meyers & Meneveauy (2010) related this length scale to the size of the vortices that are formed near the tips of the blades of the turbine. 2) The macroscale is the next scale level associated with the variations of momentum transport in the air wake near the outlet of the turbine and it is denoted as $\ell_{ma}$. 3) Finally, the megascale corresponds to the global variations of momentum from the unperturbed inlet airflow to the system outlet, where the inlet air pressure is recovered and it is denoted as $\ell_{me}$. In this way, a hierarchical system is identified under the following inequality

$$\ell_{mi} \ll \ell_{ma} \ll \ell_{me}$$ (1)

A typical characteristic of hierarchical systems is that transport processes taking place at a certain scale level determine their counterparts at a larger scale. This coupling also motivates the derivation of different classes of mathematical models that contain different amounts of information. In this way, a megascale model (i.e., the largest scale level in the system of interest here) can be derived from an upscaling process that involves an averaging step of the microscale governing equations followed by a process of systematic reduction of information (Whitaker, 2009; Wood & Valdés-Parada, 2013).

In order to have a clear view of the above, in Figure 2 different classes of the mathematical models are depicted in terms of the amount of information and the level of scale that they have. The scheme is read from left to right and top to bottom. The range of colors represents the amount of information and the characteristic lengths of the averaging regions are represented by the sizes of the circles. In this way, at the upper left corner, one can find the model with the largest amount of information and the smallest averaging region size, i.e., the continuum mechanics equations. Following the vertical blue arrow, these equations can be averaged, without any loss of information, to obtain a first type of megascale model. This model can be further simplified by means of the imposition of assumptions and restrictions (indicated by the horizontal blue arrows) in order to obtain a second type of megascale model (in orange). Note that this model is located within the zone corresponding to upscaled models, indicated by the blue rectangle. In this zone, many other upscaled models can be derived depending on the size of the averaging domain chosen and the amount of information that one is willing to conserve in the model. As a reference, the Bernoulli equation is located in the lower right extreme to identify a class of upscaled model valid at the megascale that has the least amount of information for the problem at hand. Finally, it is worth mentioning that another route towards a megascale upscaled model consists on simplifying the governing equations at the microscale and then apply an averaging step. However, for
Fig. 1: System array in the $y$ direction, including the characteristic length scales. Where $\ell_{mi}$, $\ell_{ma}$ and $\ell_{me}$ are related to microscale, macroscale and megascale, respectively. The velocity $v_{in}$ indicates the unidirectional velocity aligned with the wind turbine rotational axis.

3 Upscaling involving the actuator disc concept

This part of the work is devoted to the averaging of the total mass, linear momentum, and mechanical energy equations from the microscale up to the megascale. The analysis involves the use of the actuator disc concept, which is present in the BJ theory and it is crucial for the derivations. The analysis commences with the statement of the governing equations and boundary conditions at the microscale, followed by the averaging of each conservation equation in order to obtain the corresponding megascale counterpart. The analysis finishes with the derivation of Froude’s theorem, which states that the velocity of the actuator disc is the arithmetic mean of the inlet and outlet velocities. The demonstration of this theorem was chosen as the end point of the analysis because from it the derivation of the $16/27$ limit is straightforward. Throughout the analysis, the points of convergence with the BJ theory will be clearly identified. For the sake of simplicity and due to the fact that the theory of Joukowsky is consistent with the work presented by Betz (1927), references to this specific paper are made when the points of convergence are found. Before commencing the derivations, it is pertinent to specify that throughout this work the words ‘assumption’ and ‘restriction’ are used as two different levels of simplification of the analysis as explained by Whitaker (1988). Within this context, an assumption is less informative than a restriction because the latter is expressed in terms of a comparison between physical quantities, whereas the former is simply axiomatically imposed.

3.1 Microscale model

To commence the derivations, the following set of assumptions are imposed:

- Incompressible flow.
- Steady state transport.
Assumptions and/or restrictions

- Body forces are not included in the analysis.
- The blades and the circulating air around them are conceived as the actuator disc.

The above short list should be regarded as a set of starting assumptions; as it will be shown below, these are not the only assumptions required in the derivations. In fact, each new assumption and/or restriction will be clearly shown at the moment that it is required. Before commencing with the analysis, it is pertinent to mention that the angular momentum conservation equation is not included since it was not considered in the BJ theory. In addition, the microscale mechanical energy equation is included in the analysis, despite the fact that it is linearly dependent of the linear momentum balance equation. This is because the type of assumptions that will be imposed later for the derivation of the megascale mechanical energy equation are different from those used in the megascale linear momentum balance equation. Under these conditions, the governing microscale transport equations for mass, linear momentum, and mechanical energy can be written as follows

\begin{align}
\nabla \cdot v &= 0 \\
\rho \nabla \cdot (vv) &= \nabla \cdot T \tag{2a}
\end{align}

\begin{align}
\frac{\rho}{2} \nabla \cdot \left( \nabla v^2 \right) &= \nabla \cdot (TV) - T : \nabla v \tag{2b}
\end{align}

where \( \rho \) and \( v \) represent the density and velocity of air, respectively. In addition, the total stress tensor is defined as

\begin{equation}
T = -I p + \tau \tag{3}
\end{equation}

here \( p \) is the air pressure and the viscous stress tensor is \( \tau \). Note that, on the basis of the incompressibility assumption, the second term on the right-hand side of equation (2c) only contains the viscous term, \( i.e., \)

\begin{equation}
T^T : \nabla v = \tau : \nabla v \tag{4}
\end{equation}
here the symmetric nature of the viscous stress tensor was taken into account. This term is referred to as the viscous dissipation term (Bird et al., 2006). In the upcoming sections, the analysis is directed to the averaging of the above equations within the averaging domain \( \mathcal{V} \) sketched in figure 3a). At this point, it is worth mentioning that the shape of the averaging domain is arbitrary, in general. With this in mind, it was chosen that the averaging volume corresponds to the stream tube. However, this choice by no means implies that the analysis corresponds to the streamline equations. In fact, van Kuik (2018) carried out an analysis using a sphere as the averaging domain.

To complete the problem statement, it is necessary that sufficient boundary conditions are provided at all the surfaces of the averaging domain. These surfaces are, the inlets and outlets, the lateral surface and the parts of the wind turbine contained in \( \mathcal{V} \). The latter surfaces can be split into the moving and fixed parts as it will be detailed below.

At the entrances and exits of the averaging domain, the flow is assumed to be unidirectional and subject to the same constant pressure \( (p_1) \) without the influence of viscous stress, i.e.,

At \( \mathcal{A}_{in} \)

\[
\begin{align*}
\mathbf{n} \cdot \mathbf{\tau} &= \mathbf{0} \quad (5a) \\
\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{e}_z &= p_1 \quad (5b) \\
\mathbf{v} &= v_1 \mathbf{e}_z \quad (5c)
\end{align*}
\]

At \( \mathcal{A}_{out} \)

\[
\begin{align*}
\mathbf{n} \cdot \mathbf{\tau} &= \mathbf{0} \quad (6a) \\
-\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{e}_z &= p_1 \quad (6b) \\
\mathbf{v} &= v_2 \mathbf{e}_z \quad (6c)
\end{align*}
\]

In this work, \( \mathbf{n} \) is the unit normal vector directed from the gas phase towards the specific surface where the boundary condition is applied. Note that the inlets and outlets of the averaging domain are aligned with the \( z \)-axis (see figure 3a). In addition, the velocities \( v_1 \) and \( v_2 \) are assumed to be constants.

Directing the attention to the fixed parts of the wind turbine contained in the averaging domain, i.e., the nacelle and part of the post, the no-slip condition is applicable for the velocity and the total stress at these boundaries is denoted as \( \mathbf{T}_f \). In this way, the
corresponding boundary conditions for the velocity and stress can be written as

At $S_{M,F}$

\[
\mathbf{T} = \mathbf{T}_F \\
\mathbf{v} = \mathbf{0}
\]  

(7a)  

(7b)

Regarding the lateral surface of the averaging domain, let the normal projection of the velocity at this surface be denoted as $v_L$, which is, in general, not constant. In addition, the normal projection of viscous stress is assumed negligible with respect to pressure forces; hence

At $S_{M,L}$

\[
\mathbf{n} \cdot \mathbf{\tau} = \mathbf{0} \\
\mathbf{n} \cdot \mathbf{T} = -\mathbf{n}_pL \\
\mathbf{n} \cdot \mathbf{v} = v_L
\]  

(8a)  

(8b)  

(8c)

The moving parts of the system involve the actuator disc and the hub. In the latter, the total stress has a finite value and the normal projection of the velocity is null since no mass transport takes place at the solid-fluid interface. In this way, at this boundary ($S_{M,H}$), the following boundary conditions are imposed

At $S_{M,H}$

\[
\mathbf{T} = \mathbf{T}_H \\
\mathbf{n} \cdot \mathbf{v} = \mathbf{0}
\]  

(9a)  

(9b)

Finally, the actuator disc is conceived as a two-dimensional region in space that contains the blades of the turbine as well as the surrounding air between the blades and it could be regarded as a pseudo-continuum. The axis of this domain is aligned with the nacelle and it corresponds to the $z$-axis. This two-dimensional domain can be split into its front ($S_{M,F}$) and back ($S_{M,B}$) surfaces as sketched in figure 3b). The derivation of the boundary conditions applying to the actuator disc can be achieved by borrowing ideas of jump conditions for multiphase flow and it is presented in Appendix A. Briefly, at the front surface, the total stress is assumed to be maximum because the pressure reaches the highest value and, at the back surface, it should be minimum due to the pressure jump experienced by the fluid. Nevertheless, the maximum and minimum values of the pressure may be assumed to be of the same order of magnitude. In addition, the normal component of the velocity is denoted as $v_{oa,z}$ and has the same magnitude on both the front and back surfaces. On the basis of these assumptions, the corresponding boundary conditions at the front and back surfaces of the actuator disc are

At $S_{M,F}$

\[
\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{e}_z = T_{\text{max,rc}} \\
\mathbf{n} \cdot \mathbf{v} = v_{oa,z}
\]  

(10a)  

(10b)

At $S_{M,B}$

\[
\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{e}_z = -T_{\text{min,rc}} \\
\mathbf{n} \cdot \mathbf{v} = -v_{oa,z}
\]  

(11a)  

(11b)

Note that the areas of the surfaces $S_{M,F}$ and $S_{M,B}$ are equal and are denoted as $A_M$. In addition, $T_{\text{max,rc}}$, $T_{\text{min,rc}}$ and $v_{oa,z}$ are assumed to be position-dependent quantities.

The above equations constitute a complete description of the mass and momentum processes at the mesoscale. It is worth emphasizing that several assumptions were made in the imposition of the boundary conditions and these should be regarded as a second set of starting assumptions. In the following paragraphs, the derivation of the megascale models for each transport process is presented.

## 4 Upscaling of the mass and momentum conservation equation

To commence the upscaling process, the following averaging operator for any (scalar, vectorial or tensorial) quantity $\psi$, defined everywhere in the fluid phase (i.e., the $\beta$-phase), is introduced as

\[
\langle \phi \rangle_B = \frac{1}{V_B} \int_{\mathcal{V}_B} \psi \, dV
\]  

(12)

here $\mathcal{V}_B$ represents the domain occupied by the fluid phase within the averaging domain $\mathcal{V}$ and $V_B$ is the volume corresponding to $\mathcal{V}_B$. Following Whitaker (1999), this averaging is denoted as the intrinsic averaging operator. In addition, in many parts of the analysis is it necessary to use the surface averaging operator

\[
\langle \psi \rangle_j = \frac{1}{A_j} \int_{S_j} \psi \, dA
\]  

(13)

here $A_j = \int_{S_j} \, dA$ is the area of the surface $S_j$ ($j = \text{in}, \text{out}, \text{L}, \ldots$).

Application of the intrinsic averaging operator to equation (2a), yields

\[
\langle \nabla \cdot \mathbf{v} \rangle_B = \frac{1}{V_B} \int_{\mathcal{V}_B} \nabla \cdot \mathbf{v} \, dV = 0
\]  

(14)

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Or, after application of the divergence theorem,

\[
\frac{1}{V_\beta} \int_{\delta_{in}} \mathbf{n} \cdot \mathbf{v} \, dA + \frac{1}{V_\beta} \int_{\delta_{out}} \mathbf{n} \cdot \mathbf{v} \, dA + \int_{\delta_{L}} \mathbf{n} \cdot \mathbf{v} \, dA + \int_{\delta_{M,H}} \mathbf{n} \cdot \mathbf{v} \, dA + \int_{\delta_{M,B}} \mathbf{n} \cdot \mathbf{v} \, dA = \rho v_1 A_L \cdot \mathbf{e}_z \tag{15}
\]

where the boundary conditions at each surface were taken into account. Carrying out the straightforward simplifications and multiplying the resulting equation by \( \rho V_\beta \) leads to

\[
\rho v_1 A_{in} = \rho v_2 A_{out} + \rho \langle v_L \rangle A_L A_L \tag{16}
\]

This result simply states that the mass flow rate that enters the averaging domain can leave it from the lateral and exit boundaries. This equation can be further simplified on the basis of the following restriction for the inlet velocity

\[
\langle v_L \rangle A_L / A_{in} \ll v_1 \tag{17}
\]

This restriction is easily met whenever the magnitude of the normal component of the velocity vector is several orders of magnitude smaller than the inlet velocity \( v_1 \). In this way, the air mass will only abandon the system by passing through \( \delta_{out} \), and the final version of the megascale continuity equation is given by

\[
\rho v_1 A_{in} = \rho v_2 A_{out} = \dot{m} \tag{18}
\]

With this result at hand, the attention can now be directed to the averaging of the momentum and mechanical energy equations as shown below.

## 5 Upscaling of the linear momentum transport equation

In the problem statement at the microscopic scale, it was emphasized that the inlet and outlet velocities, as well as the rotation axis, are aligned with the \( z \)-axis. This motivates taking the projection of the vectorial equation (2b) with the axial direction marked by the unit vector \( \mathbf{e}_z \), followed by an application of the intrinsic averaging operator, in order to obtain

\[
\rho \left( \frac{1}{V_\beta} \int_{\delta_{in}} \mathbf{n} \cdot \mathbf{v} \, dA + \frac{1}{V_\beta} \int_{\delta_{out}} \mathbf{n} \cdot \mathbf{v} \, dA + \int_{\delta_{L}} \mathbf{n} \cdot \mathbf{v} \, dA + \int_{\delta_{M,H}} \mathbf{n} \cdot \mathbf{v} \, dA + \int_{\delta_{M,B}} \mathbf{n} \cdot \mathbf{v} \, dA \right) = \rho \mathbf{v}_1 \cdot \mathbf{e}_z 
\]

Application of the divergence theorem on both sides of the above equation along with multiplication by \( V_\beta \) leads to

\[
\rho \int_{\delta_{in}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{out}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{L}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{M,H}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{M,B}} \mathbf{n} \cdot \mathbf{v} \, dA = \rho \int_{\delta_{in}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{out}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{L}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{M,H}} \mathbf{n} \cdot \mathbf{v} \, dA + \rho \int_{\delta_{M,B}} \mathbf{n} \cdot \mathbf{v} \, dA
\]

where the boundary conditions have been substituted in each integral term and the inertial contribution from the fixed parts of the turbine was not written on the left-hand side of the above equation because it is null. Taking into account that \( v_1, v_2 \) and \( p_1 \) are assumed to be position-invariant and the definition of the surface-averaging operator given in equation (13), allows rewriting this last result as follows

\[
-\rho v_1^2 A_{in} + \rho v_2^2 A_{out} + \rho \langle v_L \rangle A_L = p_1 (A_{in} - A_{out}) + \rho \langle v_L \rangle A_{in} A_L = (T_{\phi,nc}) A_{L,M,F} + (T_{H,nc}) A_{L,M,B}
\]

With the aim of simplifying this result, the following set of restrictions is imposed

\[
\langle v_L \rangle / A_{in} \ll v_1 \tag{22a}
\]

\[
\langle T_{\phi,nc} \rangle A_{4,\phi} \ll \mathbf{O} \langle T_{max,nc} \rangle A_{M,F} - \langle T_{min,nc} \rangle A_{M,B} \tag{22b}
\]

\[
\langle T_{H,nc} \rangle A_{M,H} \ll \mathbf{O} \langle T_{max,nc} \rangle A_{M,F} - \langle T_{min,nc} \rangle A_{M,B} \tag{22c}
\]

\[
p_1 (A_{in} - A_{out}) \ll \mathbf{O} \langle T_{max,nc} \rangle A_{M,F} - \langle T_{min,nc} \rangle A_{M,B} \tag{22d}
\]

\[
\rho \langle v_L \rangle A_{L} \ll \mathbf{O} \langle T_{max,nc} \rangle A_{M,F} - \langle T_{min,nc} \rangle A_{M,B} \tag{22e}
\]
The first restriction is compatible with the one given in (17) and should be regarded as a restriction for the inlet velocity. The two subsequent restrictions require that the force experienced by the actuator disc is much larger than the force exerted by the wind over the fixed parts of the turbine and the hub. The final two restrictions require that the forces around the actuator disc must be much larger than the difference of forces between the inlet and outlet as well as the force over the lateral surface. Certainly, the inlet and outlet surfaces can not be equal because, from equation (18), this would lead to the conclusion that the inlet and outlet velocities are equal. Therefore, the restriction given in (22d) should be regarded as a guideline about the cross-sections of the averaging domain. These restrictions seem reasonable since the maximum values of the pressure are expected to take place near the actuator disc.

Up to this point, six restrictions have been proposed and their physical meaning has been explained. However, an alternative viewpoint can be adopted in which it is required that the size of the averaging domain is such that the boundary-layer effects are captured. Consequently, the fluid pressure at the lateral surfaces can be reasonably approximated to \( p_1 \) and therefore, the normal component of the velocity at the lateral surfaces can be assumed to be negligible with respect to \( v_1 \). Under these conditions, restrictions (22d) and (22e) are not needed because the surfaces of the averaging domain form a closed surface, as explained by van Kuik (2018). In addition, the restrictions given in (17) and (22a) are automatically satisfied. This leaves only the requirement that the forces around the actuator disc must be much larger than those experienced by the fixed parts of the turbine and by the hub.

Under these conditions, equation (21) reduces to

\[
-\left(\langle T_{\text{max,rc}}\rangle_{A_M,F} - \langle T_{\text{min,rc}}\rangle_{A_M,B}\right)A_M = m(v_1 - v_2) \tag{23}
\]

Note that the definition of the mass flow-rate given in equation (18) was used. Equation (23) is the simplified megascale version of the linear momentum equation and it corresponds to equation (20) of the work by Betz (1927), which is expressed in terms of the resistance, \( S \), that the air experiences when it flows through the wind turbine. In this way, a corollary of this analysis is that

\[
S = -\left(\langle T_{\text{max,rc}}\rangle_{A_M,F} - \langle T_{\text{min,rc}}\rangle_{A_M,B}\right)A_M \tag{24}
\]

The final element of the megascale model is the upscaled version of the mechanical energy equation.

The derivations are provided below and the resulting equation will be linked to equation (23) in order to obtain Froude’s theorem.

### 6 Upscaling of the mechanical energy equation

The last element of the BJ theory is the analysis of the mechanical energy equation. Recently, it has been proposed that this equation is crucial to determine wind farm spacing, design and power production (Kadum et al., 2019). Directing the attention to equation (2c), application of the intrinsic averaging operator leads to the following expression

\[
\frac{\rho}{2V_B} \int_{S_{in}} \nabla \cdot (\mathbf{v}^2 \mathbf{v}) \, dV = \frac{1}{V_B} \int_{S_{out}} \nabla \cdot (\mathbf{T} \cdot \mathbf{v}) \, dV - \langle \tau : \nabla \mathbf{v} \rangle \beta
\]

Note that the identity given in equation (4) was used in the viscous dissipation term. Application of the divergence theorem, taking into account the corresponding boundary conditions, yields

\[
\frac{\rho}{2V_B} \int_{S_{in}} \nabla \cdot (\mathbf{v}^2 \mathbf{v}) \, dA + \frac{\rho}{2V_B} \int_{S_{out}} \nabla \cdot (\mathbf{v}^2 \mathbf{v}) \, dA
\]

\[
+ \frac{\rho}{2V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{v}^2 \mathbf{v}) \, dA + \frac{\rho}{2V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{v}^2 \mathbf{v}) \, dA
\]

\[
+ \frac{\rho}{2V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{v}^2 \mathbf{v}) \, dA + \frac{1}{V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{T} \cdot \mathbf{v}) \, dA
\]

\[
+ \frac{1}{V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{T} \cdot \mathbf{v}) \, dA + \frac{1}{V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{T} \cdot \mathbf{v}) \, dA
\]

\[
+ \frac{1}{V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{T} \cdot \mathbf{v}) \, dA + \frac{1}{V_B} \int_{S_{in}} \mathbf{n} \cdot (\mathbf{T} \cdot \mathbf{v}) \, dA
\]

where, once more, the inertial contributions from the fixed parts and from the hub are not written on the left-hand side of the equation because they are null. In addition, the fixed parts of the turbine do not contribute to the power of the wind due to the no-slip condition. In the last two surface integrals on the right-hand side of the above result, it was assumed that at the inlets and outlets of the actuator disc the velocity is given by \( \mathbf{v} = \)
to the velocity $v_{\omega z}$. A corollary of this assumption is that the last two terms on the left-hand side of the above expression cancel each other. Furthermore, on the basis of the megascale version of the mass conservation equation given in equation (18), it follows that the first two terms on the right-hand side of the above expression also cancel each other. In this way, equation (26) can be written in the following form

$$-\frac{m}{2V_{\beta}} (v_1^2 - v_2^2) + \frac{1}{2V_{\beta}} (v_2^2 - S) = \frac{A_L}{V_{\beta}} (p_{LV}L)_{AL} + \frac{A_M}{V_{\beta}} \left( (T_{\max} v_{\omega z})_{AM,F} - (T_{\min} v_{\omega z})_{AM,B} \right) + \frac{A_{M,H}}{V_{\beta}} (n \cdot T_H \cdot v)_{AM,H} \cdot (\tau : \nabla v)^0 \quad (27)$$

This result can be simplified by imposing the following set of restrictions

$$\langle v_{LV} \rangle_{AM} \equiv \frac{A_L}{A_{in}} \ll v_1^3 \quad (28a)$$

$$A_{M,H} (n \cdot T_H \cdot v)_{AM,H} \ll A_M O \left( (T_{\max} v_{\omega z})_{AM,F} - (T_{\min} v_{\omega z})_{AM,B} \right) \quad (28b)$$

$$A_L (p_{LV}L)_{AL} \ll A_M O \left( (T_{\max} v_{\omega z})_{AM,F} - (T_{\min} v_{\omega z})_{AM,B} \right) \quad (28c)$$

$$V_{\beta} (\tau : \nabla v)^0 \ll A_M O \left( (T_{\max} v_{\omega z})_{AM,F} - (T_{\min} v_{\omega z})_{AM,B} \right) \quad (28d)$$

The first restriction is compatible to the previous restrictions for the inlet velocity given in (17) and (22a). The second restriction can be further simplified on the basis that no mass is transported from the air towards the hub, i.e.,

$$n \cdot T_H \cdot v = n \cdot \tau_H \cdot v \quad (29)$$

Therefore, only the viscous contribution should be considered on the left-hand side of (28b) and this term is quite likely to be negligible in comparison with the total power around the actuator disc.

The third restriction may be easy to meet if the size of the averaging domain is such that the boundary-layer effects are captured. As explained in the momentum transport analysis, when this is the case $v_L$ is practically null and this enables the restrictions given in (28a) and (28c) to be satisfied.

The fourth restriction translates in assuming that the viscous dissipation in the entire averaging domain is negligible with respect to the power around the actuator disc. At first sight, it would appear that neglecting viscous dissipation is a strong restriction that may fail in practice. However, since the viscous dissipation is averaged over $\delta_{\beta}$ and not only around the actuator disc, it seems plausible that this restriction is met.

At this juncture, it is pertinent to elaborate about the necessity of the actuator disc concept in the derivations. To this end, assume, for the moment, that instead of considering an actuator disc, one considers the actual geometry of the blades in the derivations. Repeating the analysis for the derivation of the megascale versions of the mass and linear momentum equations, it turns out that this change in the geometry does not translate into a modification of the final upscaled models. However, in the derivation of the megascale mechanical energy equation, there is a crucial difference: since the blades are impervious to mass transport, there is no contribution from the pressure on both $\delta_{\beta}$ and $\delta_{M,B}$. Consequently, only the viscous contribution remains and the simplifications given in (28) are not met. This is a significant difference with respect to Betz theory, where the leading term is the pressure around the wind turbine and viscous effects are disregarded in the analysis. In other words, without the actuator disc concept, the derivations provided up to this point are no longer sustained.

Returning the attention to equation (27), on the basis of the simplifications given in (28), which are reasonable for an actuator disc, allow simplifying this expression into

$$A_M \left( (T_{\max} v_{\omega z})_{AM,F} + (T_{\min} v_{\omega z})_{AM,B} \right) = \frac{m}{2} (v_1^2 - v_2^2) \quad (30)$$

In order to express this result in terms of the actuator disc velocity, it is convenient to decompose the total stress at the front and back surfaces into their corresponding surface average and spatial deviations:

$$T_{\max} = \langle T_{\max} \rangle_{AM,F} + \tilde{T}_{\max} \quad (31a)$$

$$T_{\min} = \langle T_{\min} \rangle_{AM,B} + \tilde{T}_{\min} \quad (31b)$$

Consequently, equation (30) takes the form

$$\langle v_z \rangle M A_M \left( (T_{\max} v_{\omega z})_{AM,F} + (T_{\min} v_{\omega z})_{AM,B} \right) + A_M \left( (T_{\max} v_{\omega z})_{AM,F} + (T_{\min} v_{\omega z})_{AM,B} \right) = \frac{m}{2} (v_1^2 - v_2^2) \quad (32)$$

here $\langle v_z \rangle M \equiv \langle v_{\omega z} \rangle_{AM,F} = \langle v_{\omega z} \rangle_{AM,B}$ and it corresponds to the velocity $v'$ used by Betz (1927). The above
The restriction given in (33) is satisfied whenever the total stress can be assumed to be uniform at the front and back surfaces of the actuator disc. Furthermore, if the viscous stress contribution is disregarded in the actuator disc, this restriction translates in assuming that there is a uniform pressure jump across the actuator disc, which is consistent with Froude’s theory according to van Kuik (2018).

As a final point of analysis, substitution of the upscaled linear momentum transport equation as given in equation (23), leads to obtain the following expression for the average actuator disc velocity, \( \langle v_M \rangle \).

\[
\langle v_M \rangle = \frac{v_1 + v_2}{2}
\]

which corresponds to Froude’s theorem and this concludes the analysis. At this point, it is worth noting that if both sides of the megascale momentum equation (23) are multiplied by \( \langle v_M \rangle \), the resulting equation corresponds to the mechanical energy equation (34). This correspondence is the consequence of the restrictions adopted in the derivation of the megascale models. To have more clarity about this point, let the attention be redirected to equations (21) and (27), which are the average versions of the linear momentum and mechanical energy equations before adopting the restrictions given in (22) and (28). If equation (21) is multiplied by \( \langle v_M \rangle \), the result does not correspond to equation (27). This observation is consistent with the one made by Paéz-García et al. (2017) about the averages of the linear momentum and mechanical energy equations in porous media systems.

To close this analysis about the BJ theory, it is worth pondering about the pertinence of the restrictions involved in the derivation of the megascale models. In order to obtain more insight about these restrictions, in Appendix B a numerical experiment is performed for flow conditions corresponding to a Reynolds number value, \( Re = \rho v_1/\mu = 10^7 \), with \( \ell_b = 40m \) being the length of a blade. In these simulations the actuator disc concept is not used, instead an actual wind turbine geometry is employed considering a tip speed ratio of 6 (a detailed description of the simulations is available in Appendix B). In this way, surface averages taken differentially close to the blades were computed when it was necessary to evaluate quantities averaged at the front and back surfaces of the disc. The evaluation of the restrictions shows that the most severe ones (i.e., those that differ by only one order of magnitude) are those related to mass (17), inertial linear momentum transport (22a) as well as mechanical energy transport due to inertia (28a) and stress (28c) at the lateral boundaries of the averaging domain. Certainly, these restrictions would be easier to satisfy by increasing the sizes of the averaging domain cross-sections. In contrast, the restrictions related to neglecting the contributions from the fixed parts (22b), the hub (22c), (28b) and the viscous dissipation (28d), as well as the final restriction given in (33), were easily satisfied (since a difference of at least three orders of magnitude was found). The numerical simulations showed that the pressure at the lateral surfaces corresponded to the inlet pressure. Consequently, the restrictions given in (22d) and (22e) are unnecessary because the surface integrals cancel each other as explained before. The fact that the restrictions given in (17), (22a), (28a) and (28c) were hard to satisfy is attributed to the wake effects after the turbine, which allow some mass to escape from the lateral surfaces of the averaging domain. Note that these restrictions could be easier to satisfy in shrouded wind turbines because no mass is transported from the wind towards the shroud. Directing the attention to the restrictions that were easy to meet, those related to the hub effects are satisfied due to the smaller contact area with respect to the one of the actuator disc. Furthermore, since the fixed parts are located after the actuator disc, the air pressure is considerably smaller and therefore their contribution is unimportant. Finally, since the viscous dissipation is evaluated in the entire averaging domain, it seems reasonable to conclude that its contribution is negligible.
Conclusions

In this work, the Betz-Joukowsky theory has been revisited by means of an upscaling procedure that departs from the governing equations at the microscopic scale. The motivation of this work comes from the need to identify the ancillary restrictions and assumptions that support it. In this way, the upscaling procedure used here gave rise to the corresponding equations for mass, linear momentum and mechanical energy at the megascale.

In the derivations presented in Appendix A, the actuator disc is regarded as a dividing surface that allows continuity of mass and a jump in the total stress. The latter was found to be caused by the axial surface stress exerted on the disc. Under this framework, it seems unlikely that the use of the actuator disc concept is extensible to wind turbines with moving parts that can not be reasonably compacted into a dividing surface, such as the Archimedes turbine or the vertical-axis wind turbines. This trend of thought is not in contradiction with the theory impulsed by Goldstein (1929), which requires that the turbine generates rigid trailing vortices.

Directing the attention to the remaining restrictions involved in the upscaling procedure reported here, it is worth commencing by recalling that the analysis was performed without neglecting the contributions by viscous stress in the linear momentum and mechanical energy equations. The derivation of the upscaled version of the mass conservation equation only required assuming that the mass that escapes through the averaging domain in the lateral surfaces is negligible compared to the mass that is transported axially. This restriction is in concordance with the ones required to neglect the contributions from inertial transport at the lateral boundaries (with respect to axial inertia) in the linear momentum and mechanical energy equations. These restrictions are easier to satisfy in cases in which the boundary layer effects are captured in the averaging domain. This means that the restrictions derived here can serve as a guideline for the design of the shape and size of the averaging domain. Furthermore, the derivation of the megascale linear momentum equation requires adopting additional reasonable restrictions. In specific, it is required that the force around the actuator disc is much larger than those acting over the fixed boundaries of the turbine (post and nacelle) as well as the hub and the forces acting at the inlet, outlet and lateral surfaces of the averaging domain. These forces can be shown to cancel under the assumption that pressure at the lateral surfaces can be approximated to be the inlet pressure. Finally, the derivation of the megascale mechanical energy equation requires the restriction that the power around the actuator disc is much larger than the power around the hub and the lateral surfaces. In addition, viscous dissipation effects are neglected with respect to the actuator disc power. It is worth pointing out that the restrictions presented in (28) are based upon the average of the power defined as the product of the total stress and the pointwise wind velocity. In order to recover the result from the BJ theory, it was found necessary to decompose the total stress at the surfaces of the actuator disc into their corresponding average and spatial deviations. In addition, it was found that for conditions in which there is a uniform pressure jump across the disc, the contributions from the deviations can be safely disregarded. Indeed, only after these restrictions are adopted, it is true that multiplication of the megascale linear momentum equation is equivalent to the megascale mechanical energy equation on the basis of Froude’s theorem.

The set of restrictions derived in the analysis was evaluated by means of a numerical experiment for air flow around a wind turbine. The evaluation process showed that the restrictions related to transport at the lateral surfaces of the averaging domain were the hardest to satisfy since some mass escapes from the lateral surfaces of the averaging domain. Certainly, these restrictions are easier to satisfy as the radii of the cross sections of the averaging domain are increased in order to guarantee that the boundary layer effects are captured.

The analysis presented in this work is not necessarily on the same trend of thoughts as current works, which are focused on field-scale corrections to improve the BJ theory. Nevertheless, the value of the restrictions and assumptions derived here does not only rely on the final well-known result but on the possibility of deriving more robust field-scale models. Certainly, if these potential improvements are combined with the approaches proposed in the literature such as BEM, vortex theory, among others; then more versatile and practical megascale models may be produced.
Nomenclature

\( \mathcal{A}_b \) fixed surfaces (of area \( A_b \)) of the system
\( \mathcal{A}_m \) inlet surface (of area \( A_m \)) of the system
\( \mathcal{A}_L \) lateral surfaces (of area \( A_L \)) of the system
\( \mathcal{A}_{\text{out}} \) outlet surface (of area \( A_{\text{out}} \)) of the system
\( \mathcal{A}_{M,B} \) back surface part of the moving surfaces (of area \( A_{M,B} \)) of the actuator disc
\( \mathcal{A}_{M,F} \) front surface part of the moving surfaces (of area \( A_{M,F} \)) of the actuator disc
\( \mathcal{A}_{M,H} \) hub surface (of area \( A_{M,H} \)) that belongs to the moving surfaces of the system
\( H \) curvature coefficient of a surface, m\(^{-1}\)
\( \ell_b \) length of a blade, m
\( \mathbf{l}_x \) lattice vector in the \( x \) direction, m
\( m \) mass flow rate, kg/s
\( \mathbf{n}_{FB} \) unit normal vector at the dividing surface
\( p_1 \) inlet air pressure, Pa
\( p_L \) lateral air pressure, Pa
\( r \) position vector, m
\( t \) time, s
\( T_{\text{max},nz} \) \( nz \)-component of the total stress tensor at the front surface of the actuator disc, Pa
\( T_{\text{min},nz} \) \( nz \)-component of the total stress tensor at the back surface of the actuator disc, Pa
\( T_{\phi,nz} \) \( nz \)-component of the total stress tensor at the fixed surfaces of the system, Pa
\( T_{H,nz} \) \( nz \)-component of the total stress tensor at the hub surface of the system, Pa
\( \mathbf{T} \) total stress tensor at the fixed surfaces of the system, Pa
\( \mathbf{T}_\phi \) total stress tensor at the fixed surfaces of the system, Pa
\( \mathbf{T}_H \) total stress tensor at the hub surface of the system, Pa
\( V \) averaging domain
\( \mathbf{v} \) air velocity vector, m/s
\( \mathbf{v}_1 \) air velocity at the inlet of the system, m/s
\( \mathbf{v}_2 \) air velocity at the outlet of the system, m/s
\( v_L \) air velocity at the lateral surfaces of the system, m/s
\( v_z \) \( z \)-component of the air velocity vector, m/s
\( v_s \) surface air velocity at the dividing surface, m/s
\( v_B \) air velocity at the back of the actuator disc, m/s
\( v_F \) air velocity at the front of the actuator disc, m/s
\( v_{\omega,z} \) \( z \)-component of the wake air velocity, m/s
\( V_B \) volume of the air in the averaging domain, m\(^3\)
\( \mathbf{w} \) velocity of the dividing surface, m/s

Greek symbols
\( \rho \) density of the air phase, kg/m\(^3\)
\( \rho_B \) density of the air phase at the back of the actuator disc, kg/m\(^3\)
\( \rho_F \) density of the air phase at the front of the actuator disc, kg/m\(^3\)
\( \rho_s \) surface density, kg/m\(^2\)
\( \mu \) air viscosity, Pa·s
\( \tau \) stress tensor, Pa
\( \tau_H \) stress tensor at the hub, Pa

Operator symbols
\( \langle \cdot \rangle^\phi \) intrinsic volume averaging operator in the air phase
\( \langle \cdot \rangle_j \) surface averaging operator
\( \gamma \) spatial deviations

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References


This appendix outlines the analysis of mass and linear momentum transport around the wind turbine leading to the actuator disc concept used in the main text. To this end, it is convenient to make a distinction between the air flowing in front of the turbine and the one behind it by adding the subscripts $F$ and $B$, respectively. For the sake of convenience, these two streams are regarded as two different phases and the essential transport phenomena taking place around the turbine are compacted into a dividing surface, which is the actuator disc. This approach is standard in the study of transport phenomena in multiphase systems as explained by Slattery et al. (2006). The result of compacting the information in the dividing surface is a jump boundary condition. This expression usually involves surface quantities, which can be regarded as excess quantities in the sense of Gibbs (1928) and are identified by the subscript $s$.

### A.1. Boundary condition for mass transport

For total mass transport between the $F$ and $B$ phases, the corresponding jump condition can be expressed as

$$
\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{v}_s) + 2H \rho_s (\mathbf{w} \cdot \mathbf{n}_{FB}) = \left[ \rho_F (\mathbf{v}_F - \mathbf{w}) - \rho_B (\mathbf{v}_B - \mathbf{w}) \right] \cdot \mathbf{n}_{FB} \tag{A.1}
$$

where $\rho_s$ and $\mathbf{v}_s$ represent the surface density and velocity, respectively. In addition, the surface gradient operator is defined as $\nabla_s \equiv \nabla \cdot (1 - \mathbf{n}_{FB} \mathbf{n}_{FB})$ and $H \equiv \frac{1}{2} \left( \nabla_s \cdot \mathbf{n}_{FB} \right)$ is the mean curvature of the dividing surface. Finally, the vector $\mathbf{w}$ denotes the displacement of the dividing surface. For the case at hand, the following assumptions can be immediately proposed,

1. There is no surface accumulation because the analysis is carried out under steady state conditions.

2. Since the axis of the actuator disc is only moving in the angular direction, it follows that $\mathbf{n}_{FB} \cdot \mathbf{w} = 0$

Consequently, equation (A.1) reduces to

$$
\nabla_s \cdot (\rho_s \mathbf{v}_s) = \mathbf{n}_{FB} \cdot (\rho_F \mathbf{v}_F - \rho_B \mathbf{v}_B) \tag{A.2}
$$

Furthermore, the surface density is defined as

$$
\int_{A_{MF}} \rho_s \, dA = \int_{\tilde{\gamma}_F} (\rho - \rho_F) \, dV + \int_{\tilde{\gamma}_B} (\rho - \rho_B) \, dV \tag{A.3}
$$

where $A_{MF}, \tilde{\gamma}_F$ and $\tilde{\gamma}_B$ represent the space occupied by the dividing surface and by the fluid in the front and behind the surface, respectively. Restricting the analysis to situations in which the mass passing by the actuator disc is negligible compared to the order of magnitude of the mass entering and leaving it, i.e.,

$$
\int_{A_{MF}} \rho_s \, dA \ll \int_{\tilde{\gamma}_F} \rho_F \, dV + \int_{\tilde{\gamma}_B} \rho_B \, dV \tag{A.4}
$$

it follows that $\rho \approx \rho_F = \rho_B$ and equation (A.2) reduces to

$$
\mathbf{n}_{FB} \cdot \mathbf{v}_F = \mathbf{n}_{FB} \cdot \mathbf{v}_B = v_{\omega z} \tag{A.5}
$$
which translates into equations (10b) and (11b) of the main text.

A.2 Boundary condition for momentum transport

For linear momentum transport, the corresponding jump boundary condition can be expressed as

$$\frac{\partial \rho_s \mathbf{v}_s}{\partial t} + \nabla_s \cdot (\rho_s \mathbf{v}_s \mathbf{v}_s) + 2H \rho_s \mathbf{v}_s (\mathbf{w} \cdot \mathbf{n}_{FB}) = n_{FB} \cdot [\rho_F \mathbf{v}_F (\mathbf{v}_F - \mathbf{w}) - \rho_B \mathbf{v}_B (\mathbf{v}_B - \mathbf{w})] + \rho_s \mathbf{g}
+ (\nabla_s \cdot \mathbf{T}_s + 2H n_{FB} \cdot \mathbf{T}_s) - n_{FB} \cdot (\mathbf{T}_F - \mathbf{T}_B)
$$

(A.6)

This equation is readily simplified, on the basis of the assumptions adopted above for mass transport, into

$$n_{FB} \cdot (\mathbf{T}_F - \mathbf{T}_B) = (n_{FB} \cdot \mathbf{v}_F) \cdot \mathbf{v}_F + \nabla_s \cdot \mathbf{T}_F + 2H n_{FB} \cdot \mathbf{T}_F
$$

(A.7)

The axial component of this expression is obtained by taking the inner product on both sides of the equation by the unit vector $\mathbf{e}_z$

$$n_{FB} \cdot (\mathbf{T}_F - \mathbf{T}_B) \cdot \mathbf{e}_z = \nabla_s \cdot \mathbf{T}_F \cdot \mathbf{e}_z + 2H n_{FB} \cdot \mathbf{T}_F \cdot \mathbf{e}_z
$$

(A.8)

This result states that the axial stress around the actuator disc is proportional to the axial surface stress of the disc, which is intuitively appealing. As a matter of fact, the stress exerted on the front surface of the disc must be maximum and the one applying at the back face is minimum and they are unequal according to equation (A.8). This justifies the boundary conditions given in equations (10a) and (11a) of the main text.

Appendix B: Numerical evaluation of the restrictions imposed in the actuator disc analysis

The analysis presented in the main body of the work is general in the sense that it is not restricted to a particular system or turbine geometry. The restrictions made through the derivation of the megascale models are expressed in terms of inequalities that ponder magnitude orders of different quantities. Furthermore, a quantitative idea about the meaning of these estimates is desirable in order to gain clarity for practical applications at the expense of losing generality. This motivates performing a numerical experiment over a particular realization of the system for a specific wind turbine geometry by solving the mass and momentum equations presented in Section 3.1. Certainly, the numerical simulations could have been performed using an actuator disc; however, the tests presented here with a realistic airfoil profile (as the one sketched in Figure B.1) are more exigent and closer to experimental validation.

First, it is necessary to specify the boundary conditions at the lateral surfaces ($\mathcal{S}_l$) as well as at the top ($\mathcal{S}_T$) and bottom ($\mathcal{S}_B$) of the system so that the turbine is unconfined (a recent study of unconfined turbines is available from Gauthier et al. (2016)). Assuming that the turbine is inserted in a parallel array, it is reasonable to impose the following periodic condition at the lateral surfaces

$$\mathcal{S}_l, \quad \mathbf{v}(\mathbf{r}) = \mathbf{v}(\mathbf{r} + \mathbf{l}_3)
$$

(B.1)

with $\mathbf{r}$ and $\mathbf{l}_3$ being a position vector and the lattice vector in the $x$-direction, respectively. At the top boundary, it is assumed that the viscous effects are no longer relevant in the total stress tensor and therefore the following boundary condition is imposed

$$\mathcal{S}_T, \quad \mathbf{n} \cdot \mathbf{\tau} = 0
$$

(B.2)

Finally, at the bottom surfaces the non-slip condition applies

$$\mathcal{S}_B, \quad \mathbf{v} = \mathbf{0}
$$

(B.3)

The numerical solution is carried out using the commercial finite element solver Comsol Multiphysics 5.3 using a rotating machinery module under a frozen rotor approach with wall distance initialization. The flow conditions of interest correspond to a Reynolds number, $Re = \rho \mathbf{v}_t (\ell_b) / \mu = 10^7$, with $\ell_b = 40 m$ being the length of a blade and a tip-speed-ratio (TSR= velocity at the blade tip/inflow velocity) of 6, which is a reasonable representation of a high-power wind turbine that is moving under stable conditions. The transport equations (in a weak formulation) were solved in a fully coupled manner with the algebraic multigrid solver included in the software. In addition, custom mesh convergence analyses were performed in order to guarantee the independence of the results with this numerical parameter. The height and width of the system were chosen to be $10 \ell_b$ and its length was determined to be $60 \ell_b$ with the turbine located at $15 \ell_b$. This configuration was considered sufficient to guarantee that the outlet pressure corresponds to the one at the inlet.
Since the Reynolds number value corresponds to turbulent flow, a RANS model is used in the numerical solution. Due to the accuracy near the walls and far from them, the SST model was used. In order to test the validity of the SST model, the numerical predictions are compared to those resulting from solving the Navier-Stokes equations. Unfortunately, this comparison was only possible for some specific combinations of TSR and Re number values below the first Hopf bifurcation. In Figure B.2, the comparison of the predictions resulting from taking $Re = 10^5$ (TSR = 6), $Re = 10^6$ (TSR = 3) and $Re = 10^7$ (TSR = 1), is presented in terms of the axial component of the velocity vector. Evidently, excellent agreement (below 1% of relative error) is found between both numerical solutions and this serves as a validation of the RANS solution approach used here.

Directing the attention to the geometry of the averaging domain, its shape and size must be such that it captures the essential features of the transport process under study. According to the analysis presented in the main body of the paper, the outlet boundary has to be larger than the inlet in order for the outlet velocity to be smaller than $v_1$ as implied from equation (18). Therefore, as a first approach, the inlet surface was set to correspond to a circle of radius equal to $\ell_b$, and a cylinder was drawn by extruding this surface over a distance of $14.7\ell_b$. Subsequently, a frustum of a right circular cone of height $2\ell_b$ and a ratio of the largest to the smaller radii of 1.2, was drawn. Finally, the largest cross-section of this geometrical entity was extruded over a distance of $43.3\ell_b$. In this way, the radius of the outlet circle turned out to be $1.27\ell_b$. It should be emphasized that this choice of shape and size of the averaging domain is a first approach that was proposed to examine the restrictions derived in the analysis. Certainly, other configurations can be considered.

In Figure B.3 an example of the distribution of the velocity in the system and in the averaging domain is shown normalized with the inlet velocity. As expected, the maximum velocities are found near the tip of the turbine blades (see Figure B.3a); in addition, the downwind wake appears to be reasonably captured within the averaging domain (Figure B.3b). This observation is corroborated with the velocity magnitude profiles shown in Figure B.3c). Note that in this figure the range of values was modified with the purpose of appreciating the spatial variations of the velocity.

With the system and averaging domain dimensions set in place, it is possible to analyze the numerical results and ponder about the pertinence of the restrictions involved in the BJ theory. To this end, it was first verified that $p_L$ corresponded to $p_1$; in addition, the product $n \cdot v$ at the lateral surfaces was found to be four orders of magnitude smaller than the inlet velocity. Second, all the integrals involved
Fig. B.2: Comparison of the predictions of the profiles of the axial component of the dimensionless velocity resulting from the numerical solution of the Navier-Stokes equations and the RANS model for three values of the Reynolds number and tip-speed ratios (TSR). The location of the turbine is sketched in each plot.

in the inequalities corresponding to mass (17), linear momentum (22) and mechanical energy [(28) and (33)] were computed. In particular, the integrals associated with the actuator disc were obtained by taking cross-sections of the averaging domain at locations that were sufficiently close to the front and back of the turbine blades. It was found that the hardest restrictions (i.e., those that only maintained one order of magnitude of difference) were the following: the restrictions related to mass (17); inertial momentum (22a); inertial (28a) and stress (28c) mechanical energy transport through the lateral surfaces of the averaging domain. In addition, since \( p_L \) was found to be approximately equal to \( p_1 \), it follows that the inequalities given in (22d) and (22e) are no longer necessary because the sum of the integrals at the inlets, outlets, and lateral surfaces form a closed surface. The remaining restrictions dealing with the stress applied to the fixed surfaces and the hub, as well as by the hub power and viscous dissipation, were easily satisfied by, at least, three orders of magnitude of differences. The last restriction involved in the derivations deals with the pertinence of approximating the total stress in the actuator disc by its average value (33). This restriction was found to be met by two orders of magnitude of difference and it can thus be considered to be satisfied. As a final comment, the fact that the largest differences were found in the evaluation of properties at the lateral surfaces may be attributed to 1) the length of the averaging domain is quite large and this complicates meeting these restrictions or 2) these restrictions are inherently hard to meet by the BJ theory since some mass escapes these surfaces.
Fig. B.3: Views of the normalized velocity in the numerical experiment. a) Profile view zoom near the turbine blades including a cut-plane of the velocity field magnitude, b) Back view of the system with the streamlines, c) Lateral view of the entire system with a cut-plane of the velocity field magnitude.